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STABILIZATION AND EXTREMAL PROPERTIES OF RESONANT MODES OF BIPEDAL LOCOMOTION*

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The resonance property of anthropomorphic locomotion is investigated. (A form of locomotion is said to be resonant (periodic) if all the elements of the walking device oscillate at commensurable frequencies**). A control law is demonstrated under which a periodic nominal mode is non-asymptotically stable. The energy requirements of periodic and neighbouring non-periodic modes are compared. It is shown that the energy-consumption functional has a local minimum on stable resonant modes.

1. Statement of the problem. Locomotion as a resonance phenomenon. It is an obvious fact that regular locomotion is resonant: all elements of the walking device oscillate "in time" with the locomotion, so that their frequencies are commensurable with the step frequency. Many studies of the process of bipedal locomotion /1-4/ have been concerned with the description of regular, periodic locomotion. Nevertheless, these studies have not been able to observe certain specific effects. Indeed, resonant and non-resonant motions can be distinguished only by actual comparison. It follows that periodic locomotion must be studied together with its non-periodic neighbourhood.

A specific resonance effect is, for example, the property of stable periodic motions to maximize or minimize a fairly large class of functionals /5-8/. In particular, it is a reasonable assumption that the energy requirements of locomotion are minimized by well-organized stable periodic modes. To verify this assumption we shall construct a simple model of periodic locomotion which is stable in a certain sense, and compare the values assumed by the energy functional in periodic and nearby non-periodic walking modes.

2. Description of the model. Fundamental equations. Consider a bipedal walking device consisting of a body and two weightless articulated legs, each made of two sections, attached by a hinge to the body at a point O (Fig.1). It is assumed that the device is supported at any specific time by only one leg, without impact; each leg touches the supporting surface at a single point; the constraint is unilateral.

Let us assume that the device is moving in the NXZ plane (Fig.1). We shall use the following notation: ρ is the distance from O to centre of mass C of body; M is the mass, J is the moment of inertia of the body about O; X, Z are the coordinates of O; X_v, Z_v are the coordinates of the support point; θ is the angle between the vector OC and the positive direction of the NZ axis (Fig.1).

We define the following non-dimensional variables:

$$\tau = \sqrt{\frac{M\rho g}{J}}t, \quad j = \frac{J}{M\rho^2}, \quad x = \frac{X}{\rho}, \quad z = \frac{Z}{\rho}, \quad x_v = \frac{X_v}{\rho}, \quad z_v = \frac{Z_v}{\rho} \tag{2.1}$$

The derivative of the angle θ with respect to non-dimensional time τ will be denoted by θ'.

The body oscillates in the plane, obeying the following equation in terms of the variables (2.1) /4/:

$$\begin{gathered}(j + (z - z_v)\cos\theta + (x - x_v)\sin\theta)\,\theta'' + ((x - x_v)\cos\theta + \\ (z - z_v)\sin\theta)\,\theta'^2 - (j + z'')\sin\theta + x''\cos\theta = (j + z'')(x - x_v) - \\ x''(z - z_v)\end{gathered} \tag{2.2}$$

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The non-dimensional coordinates x_ν, z_ν of the points at which the extremities are placed are piecewise-constant functions of time, so that the functions $x - x_\nu$, $z - z_\nu$ occurring in Eq.(2.2) are discontinuous (piecewise-continuous).

We shall assume that the device is subject to some control system which is capable of driving the suspension point of the legs according to any prescribed sufficiently smooth law of motion $x(\tau)$, $z(\tau)$. According to the semi-inverse method of given synergy /1/, the function of the control can be fulfilled by the trajectory, the velocity or the acceleration of this point.

We consider regular locomotion along a horizontal straight line with a step of duration τ_0 and length l:

$$x_\nu = (\nu - 1)\, l,\ z_\nu = 0,\ \tau \in [(\nu - 1)\,\tau_0,\ \nu\tau_0]\ \ \nu = 1,\ 2,\ \ldots \tag{2.3}$$

The time τ_0 is the fundamental period of the process. Let us suppose that the point of attachment of the legs is moving in accordance with a given τ_0-periodic law, while the law of motion of the body has to be determined by integrating Eq.(2.2). We shall say that the locomotion is periodic or resonant if the body oscillates an integral number of times over an integral number of steps: $\theta(\tau + \tau_K) = \theta(\tau)$, where $\tau_K = \text{const}$ is the period of the body's oscillations, $n\tau_0 = m\tau_K$, $m, n \in N^+$.

The average power consumed per unit path in N steps is measured by the following functional /4/:

$$w_N = \frac{1}{Nl} \int_0^{N\tau_0} (|q(\theta' - \alpha')| + |u(\alpha' - \beta')|)\, d\tau \tag{2.4}$$

Here q, u are the controlling torques at the hip and knee of the supporting leg and α and β are the angles between the thigh and shin, respectively, and the vertical.

In our model the controlling torques q, u as well as the angles α and β, can be evaluated by closed formulae in terms of the functions $x(\tau)$, $z(\tau)$ and $\theta(\tau)$ and their derivatives with respect to time /4/.

3. *Nominal mode.* The fundamental mode will be a periodic "gait" in which the body oscillates once for each walking step $n = m = 1$.

For example, let us consider "comfortable"locomotion/4, 9, 10/, in which $\tau_K = \tau_0$ and the point of attachment of the legs moves at a constant height h relative to the supporting surface and at a constant velocity $v = l/\tau_0$. If we assume that the step is symmetric $(x(0) = -l/2)$, the motion of the point O is governed by the law

$$x = x_* = v\tau - l/2 \quad z = z_* = h \tag{3.1}$$

and Eq.(2.2) becomes

$$((j + h\cos\theta) + (x_* - x_\nu)\sin\theta)\,\theta'' + ((x_* - x_\nu)\cos\theta - h\sin\theta)\,\theta'^2 - j\sin\theta = j\,(x_* - x_\nu)$$

Under conditions (2.3) and (3.1) this equation has a continuous τ_0-periodic solution $\theta_*(\tau)$. The comfortable mode $x_*(\tau), z_*(\tau), \theta_*(\tau)$ will be considered as the nominal mode.

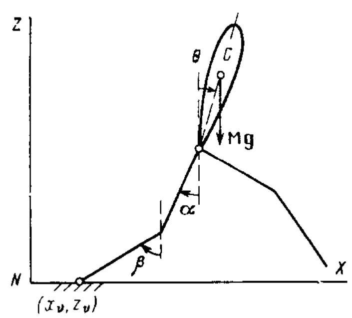

Fig.1

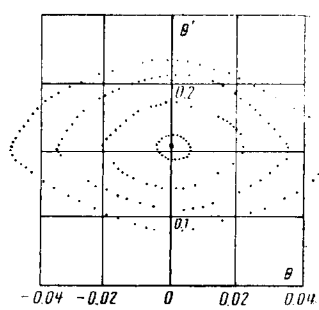

Fig.2

4. *The stabilization problem.* It is well-known that the motion $\theta_*(\tau)$ of the body in comfortable lomotion is unstable /4/, and controlling torques at the joints cannot ensure the existence of near-nominal non-periodic walks. To obtain a family of walks with this property, we must construct an algorithm that will stabilize comfortable locomotion.

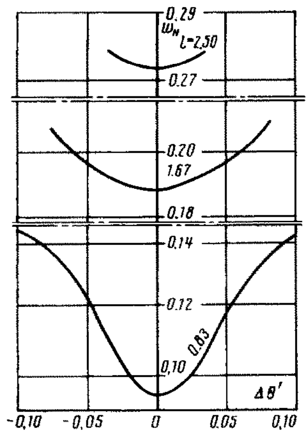

Fig.3

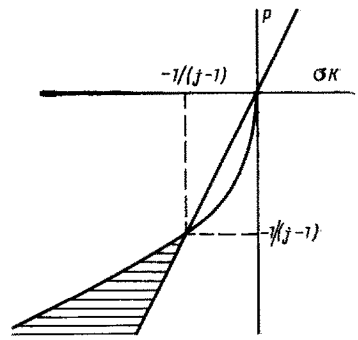

Fig.4

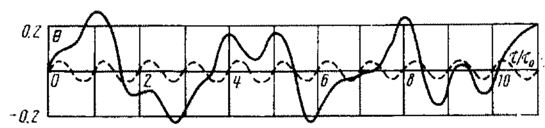

Fig.5

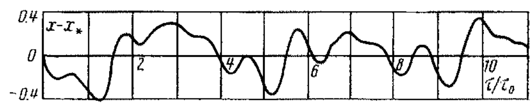

Fig.6

Various processes of free stabilization can be accomplished, e.g., by regulating the length and duration of the step /4/. In this paper we shall solve a different problem: to create stable τ_0-periodic oscillations of the body in periodic locomotion with fixed step length l and fixed step duration τ_0. The problem may be formulated as follows.

Let the oscillations of the body be described by Eq.(2.2) and assume that the footprints (2.3) are given. It is required to find a control law of the walking device such that: 1) a comfortable motion (3.1) of the point of attachment of the legs is possible; 2) Eq.(2.2) has a τ_0-periodic solution $\theta = \theta_*(\tau)$ in the special case (3.1); (3) the particular solution $\theta_*(\tau)$ is stable in some sense.

5. ***Stabilization of the body (the limiting problem).*** Let us consider Eq.(2.2) on the assumption that the control is the acceleration of the point of attachement. Define

$$x'' = \sigma \operatorname{arctg} k(\theta - \theta_*), \ z = h \tag{5.1}$$

If there is no controlling torque at the hip-joint, the body of the device may be treated as a physical pendulum on a mobile base. If the point of attachement remains at a constant height $(z = h)$, the unstable equilibrium $(\theta = 0)$ of the pendulum may be stabilized by subjecting the coordinate x to the control law $x'' = \sigma \operatorname{sign} \theta$, $\sigma = \text{const}$, $\sigma > 0$. This simple piecewise-constant control law enables us to give an analytical treatment of the stabilization of the pendulum (see our preprint cited in the footnote at the beginning of this paper). The attempt to apply a similar control in our case, where we have to stabilize the body in locomotion (Sect.4), obliged us to replace the discontinuous function $\operatorname{sign}(\theta - \theta_*)$ by the continuous function $\operatorname{arctg} k(\theta - \theta_*)$, $k = \text{const}$, which is better suited to numerical manipulation.

The system of Eqs.(2.2) and (5.1) has a particular solution $x = x_*$, $\theta = \theta_*$. In the general case, horizontal motion of the point is non-comfortable and the law $x(\tau)$ is determined by simultaneous integration of Eqs.(2.2) and (5.1).

As a first approximation to the solution of system (2.2), (5.1), let us consider the following limiting problems. Suppose that the motion of the point O is comfortable, $x(\tau)$ in Eq.(2.2) is defined by (3.1), but the body is subjected to the control $x''(\tau)$, where $x''(\tau)$ in (2.2) is defined by (5.1). (The equations of the limiting problem constitute an asymptotic form of system (2.2), (5.1), in the case of a body of small dimensions; see our preprint cited above). Eq.(2.2) in the limiting problem is decoupled from (5.1) and becomes an equation with τ_0-periodic piecewise-continuous coefficients, which has a τ_0-periodic solution $\theta = \theta_* (\tau)$. This solution will now be investigated for stability.

We will first consider the case in which the footprints (2.3) determine not locomotion but standstill: $l = 0$, $x_v = 0$. The corresponding comfortable mode is $x_* (\tau) = 0$, $z_* (\tau) = h$, $\theta_* (\tau) = 0$. Linearizing Eq.(2.2) according to the assumptions of the limiting problem in the neighbourhood of $\theta_* = 0$: $\theta'' = j (1 - \sigma k (h + 1)) \theta/(j + h)$, we obtain the necessary conditions for the body to be stabilized at a standstill:

$$\sigma k > 1/(1 + h) \tag{5.2}$$

The stabilizing effect of the control (5.1), (5.2) in the limiting problem with non-zero step length has been verified numerically. The computations were run for a device with the following anthropomorphic data: $M = 70$ kg, $J = 9.63$ kg$\cdot$m^2, $\rho = 0.24$ m, the length of each element in the aritculated leg 0.425 m; the dimensional parameters of nominal mode are $H = \rho h = 0.8$ m; $t_0 = \tau_0 [J/(M\rho h)]^{1/2} = 0.625$ sec. Numerical integration of the limiting locomotion problem with non-dimensional step length 1.67 and control parameters $\sigma = 0.1$, $k = 100$ yielded the results shown in Fig.2, which shows the pattern of point transformations in the phase space (θ, θ') in one period τ_0. The fixed point of the transformation is that corresponding to periodic motion $\theta_* (\tau)$. All the point transformations lie on ellipsoidal curves, demonstrating the stability of the nominal mode. The different curves correspond to different deviations $\Delta\theta' (0)$ from the initial data $\theta_*' (0)$ of the periodic mode on the assumption that $\Delta\theta (0) = 0$.

A general analysis of the computer treatment of the limiting problem has shown that the control (5.1), (5.2) can stabilize the nominal mode if the non-dimensional step length l remains less than a certain critical value $l_0 (\sigma, k)$. For any $l, 0 \leqslant l < l_0 (\sigma, k)$ there is a neighbourhood of the point $(\theta_*(0), \theta_*'(0))$ in the phase plane (θ, θ') which generates a family of near-nominal non-periodic gaits (beyond the boundary of this neighbourhood feedback breaks down /11/ and the device falls). A few specific values of the energy functional (2.4) were computed for nominal periodic and nearby non-periodic modes; the number N of steps over which the functional was averaged was taken to be so large that the functional was independent of N to within the given accuracy.

Fig.3 shows some plots of the energy functional w_N for nearby non-periodic modes against the initial angular velocity mismatch $\Delta\theta' = \theta' (0) - \theta_*' (0)$ of the body. Figs.2 and 3 together clearly reflect the expected result: the energy required by locomotion is a minimum in a stable resonant mode.

6. Stabilization of locomotion. We now consider the general problem (2.2), (5.1). It can be shown that the control (5.1), (5.2), which stabilizes the motion of the body in the limiting problem, does not do so in the full problem of locomotion. The control (5.1) does not include feedback from the deviation of the body's forward motion from the nominal mode $x_* (\tau)$ i.e., the forward motion is not stabilized. This causes instability both of forward motion and of the body's oscillations. We shall refer to simultaneous stabilization of both the forward and oscillatory motion of the body as stabilization of locomotion.

To achieve such stabilization, we add a term to the control to represent feedback of the deviation $x - x_*$ from the nominal forward motion. Instead of (5.1), therefore, we have a new law of motion:

$$x'' = \sigma \operatorname{arctg} k (\theta - \theta_*) + p (x - x_*), \ p = \text{const}, \ z = h \tag{6.1}$$

Let us investigate system (2.2), (6.1) for stability. Let $l = 0$, $x_v = 0$; then $x_* (\tau) \equiv 0$, $\theta_* (\tau) \equiv 0$. Linearizing the system in the neighbourhood of x_*, θ_*, we obtain

$$\begin{gathered} \theta'' = j (j + h)^{-1} ((1 - \sigma k (h + 1)) \theta + (1 - (1 + h) px) \\ x'' = j (\sigma k\theta + px) \end{gathered} \tag{6.2}$$

The conditions for the stability of this linear system can be written in the form

$$\begin{gathered} \sigma k < p < (1 + \sigma k (h + 1) - 2\sqrt{\sigma k (1 - j)})/(j + h) \\ \sigma k < 1/(1 - j) \end{gathered} \tag{6.3}$$

The parameters σ, k, p satisfying these inequalities will guarantee non-asymptotic stability of the linear system in the case of standstill.

Fig.4 shows the parameter plane $(\sigma k, p)$ with the region (6.3) hatched. Note that according to (6.3) and Fig.4, any motion with $p = 0$, i.e., with no feedback from forward motion in the control, is unstable.

In our investigation of the complete problem of locomotion, described by Eqs.(2.2) and (6.1), we selected control parameters which satisfied the necessary conditions (6.3) for stability in a standing position. Numerical integration of system (2.2), (6.1) subject to conditions (6.3) showed that over a certain set of values of l, $0 \leqslant l \leqslant l_0 (\sigma, k, p)$, the control

(6.1) does stabilize the motion in a certain sense. Over a long time interval $[0, N\tau_0]$, where $N > 50$ (in our computations N was chosen so as to give the functional w_N a stationary value), the perturbed motion $\theta(\tau)$, $x(\tau)$ remained in a bounded neighbourhood of the comfortable mode $\theta_*(\tau)$, $x_*(\tau)$, which is small together with the perturbation.

Figs.5 and 6 illustrate different types of behaviour of the functions $\theta(\tau)$, $x(\tau) - x_*(\tau)$ in perturbed motion. The non-periodic mode depicted here corresponds to control parameters $\sigma = 10, k = 1, p = -9.4$, a step length $l = 2.50$ and a perturbation $\Delta\theta' = 0.07$. The curve $\theta_*(\tau)$ is shown for comparison. It is obvious that the perturbed motion follows a pattern similar to the nominal one, with no systematic departures from nominal over this particular time interval.

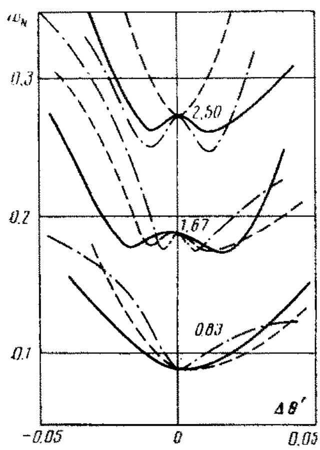

Fig.7

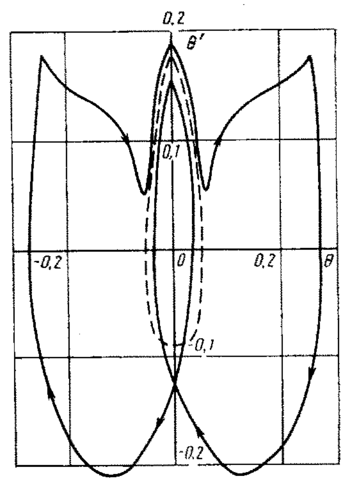

Fig.8

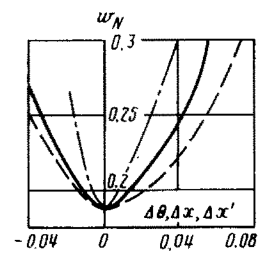

Fig.9

Thus, subject to conditions (6.3), the control (6.1) generates a family of non-periodic motions which remain close to nominal periodic locomotion over a fairly long time interval.

*7. **Extremal properties of resonant modes of locomotion.*** Fig.7 shows the results of a numerical investigation of the energy requirements of nominal periodic modes and their non-periodic neighbourhoods. The functional w_N is plotted against the initial perturbations of the angular velocity $\Delta\theta'$, on the assumption that the other phase coordinates are unperturbed: the solid curves are drawn for parameter values $\sigma = 10$, $k = -1$ and $p = 9.4$, the dashed curves are for $\sigma = 5$, $k = -1$ and $p = 4.9$, and the dash-dot curves are for $\sigma = 15$, $k = -1$ and $p = 13.9$. The figures in the plot indicate the appropriate non-dimensional step lengths l (the dimensional lengths corresponding to $l = 0.83 \cdots 2.50$ are $L = 0.2 \ldots 0.6$ m). In all cases w_N, plotted against $\Delta\theta'$, has a local extremum at the nominal periodic mode, i.e., at $\Delta\theta' = 0$. For small step lengths w_N is a minimum at $\Delta\theta' = 0$. In that case τ_0-periodic (resonant) locomotion is energetically preferable to nearby non-resonant modes.

As the step length is increased, the minimum may bifurcate: the value of w_N at $\Delta\theta' = 0$ becomes a local maximum, and two local minima of w_N appear in the neighbourhood of the nominal, at certain values of $\Delta\theta' \neq 0$.

The appearance of the local minimum of w_N is a consequence of a known fact: comfortable modes of locomotion are energetically inferior to non-comfortable modes /4, 12, 13/. It is interesting that the new local minima of w_N are achieved at new periodic and stable modes of locomotion.

Fig.8 is a phase portrait of the oscillations of the body in one of these modes (which turned out to be $4\tau_0$-periodic; parameters: $\sigma = 10$, $k = -1$, $p = 9.4$, $l = 2.09$). The body performs one oscillation in four walking steps (the cusps in Fig.8 represent the times at which the body shifts from one leg to the other). For comparison, the phase portrait of a nominal comfortable τ_0-periodic mode is also shown (the dashed curve). The two local minima of w_N in the plot correspond to the same mode, but phase-shifted.

Thus, based on the extremal property of the energy functional, one obtains new periodic modes without prior periodicity assumptions.

It should be noted that the local minimum of w_N is achieved simultaneously with respect to all phase variables. Fig.9, for example, shows plots of w_N against the deviations $\Delta\theta$, Δx, $\Delta x'$ (the solid, dashed and dash-dot curves, respectively) at $\Delta\theta' = \Delta\theta_{01}'$, where $\Delta\theta_{01}'$ corresponds to the above-mentioned $4\tau_0$-periodic motion.

On the basis of the above investigation, it is reasonable to postulate that the energy requirements of locomotion have local minima at stable periodic (resonant) modes. This may well be at least one of the reasons for the observed resonance properties of bipedal locomotion.

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